RUDOLF R.H. DITTRICH'S INTERNET PUBLICATION SERIES ON ENGINEERING AND PHYSICAL TOPICS

VOLUME 2008-0

PRELIMINARY FREE DOWNLOAD VERSION

CONTAINS VERSION 1.1 OF THE FOLLOWING ELEMENTARY ARTICLE:
"Elementary Remarks on the Qualitative Modeling of Some Simplified Systems"

RUDOLF R.H. DITTRICH'S INTERNET PUBLICATION SERIES
ON ENGINEERING AND PHYSICAL TOPICS, VOLUME 2008-0

Table of Contents

INTRODUCTION

The article contained in this document represents the contents of the elementary article that was testwise available on the author's web page

`http://www.rudolf-rh-dittrich.com/rdmisc.html`

since October 2007.

In the future, the contents of the web page may change on short notice and need not, therefore, correspond to the present article.

DISCLAIMER & COPYRIGHT INFORMATION:

In order to obtain copies, or any other information pertaining to this article, you are requested to write to the author's address given on page 2 of this document.

Original contents of author's 2007 webpage

http://www.rudolf-rh-dittrich.com/rdmisc.html

follows on next pages

Elementary Remarks on the Qualitative Modeling of Some Simplified Systems

(Download link available at end of article)

1 Overview

The following short article is not meant to present any new material, possibly except for the notion of *"qualitative"* modeling which is used by the author in the sense of describing the approximate modeling of systems whose physical laws and properties have been replaced by a set of suitably chosen model properties. Instead of presenting new computational methods this is the first in a future series of articles presenting some elementary relations for those wishing to apply basic computational techniques to real problems without having any background in computational physics.

Prior to performing a numerical simulation or an analytical description of some engineering systems one may occasionally approximate both the system and its evolution in time in a qualitative manner that involves the characteristic function of the individual subparts of the system. Considering a very simplified example problem, the qualitative description using such approximations is given. The article closes with a remark on the next articles where these techniques will be applied to more complex systems.

2 Disclaimer

This article may contain unintentional mistakes like typing errors, calculation errors and other flaws. It is a common experience that such errors are possible, even if the utmost care is taken to prevent them. The author therefore disclaims all warranties, either implied or express, relating to the accuracy of the information contained in this article or its sequels. You are therefore kindly requested to inform the author of any mistakes in order to facilitate their due correction in future editions. The author does not allow the incorporation of this article, or parts of it, into other documents, web sites, or any other papers. Furthermore, no permission is granted to distribute this document electronically, be it via electronic mail or any other imaginable means, or to store it, or part of it, in a computer or any other data-retrieval system with the sole exemption of a computer's browser cache which temporarily stores previously selected web site content. Any commercial use of this article, its contents, or the results of its contents' application to commercial products, including the distribution, copying and selling either by individuals, companies, or other parties, is not and will not be allowed without first having negotiated with the author the financial terms and legal conditions for which you will be granted the non-exclusive right to use the beforementioned contents. This applies even if the non-private use were non-profit oriented. Any violation of the aforementioned rules constitutes an infringement of the author's copyright.

3 Some Elementary Approximating Functions

In many engineering systems, the distribution of mass, energy, momentum $\rho(\vec{x},t)$, $\varepsilon(\vec{x},t)$, $\vec{g}(\vec{x},t)$ and other properties $P(\vec{x},t)$ may often be described in a simplified manner using the characteristic function of the systems' individual parts (definition of the characteristic function: see [1], p. 1). Thus the kinematics of a non-deformable non-rotating machine part with homogeneous mass density, ρ_0, is given by $\rho(\vec{x},t) = \rho_0 \chi_{G(\vec{x}-\vec{v}t)}$ if $\chi_{G(\vec{x})}$ denotes the characteristic function of the part at time $t = 0$. In case of a system composed of N disjoint parts described by point sets G_j, each of uniform mass density ρ_j, the characteristic functions χ_{G_j} suffice to describe the distribution of mass as follows: $\rho(\vec{x}) = \sum_j \rho_j \chi_{G_j}(\vec{x})$ $(1 \leq j \leq N)$. Similar relations hold for other properties. Recall that the characteristic function $\chi_G(\vec{x})$ equals 1 for all $\vec{x} \in G$ and 0 if $\vec{x}$ is outside of G and that it can usually be computed for any domain. For simplicity's sake, the present article will consider rectangular and triangular domains only which can easily be extended to more complex geometries using appropriately chosen diffeomorphisms. We are going to qualitatively describe the geometry of individual building blocks and system subparts by their characteristic function corresponding to the subset of $\mathbb{R}^3$ the parts belong to. Recall that the characteristic (rectangle) function of a real interval $[a,b] \subset \mathbb{R}$, denoted by $\chi_{[a,b]} : \mathbb{R} \to [0,1] \subset \mathbb{R}$, can be described making use of Heaviside's step function Θ (disregarding complications at the interval's boundaries): $x \mapsto \chi_{[a,b]}(x) := \Theta(x-a) - \Theta(x-b) = \begin{cases} 1 & \text{for } x \in [a,b] \\ 0 & \text{for } x \in \mathbb{R}\backslash[a,b] \end{cases}$ This may be approximated by the well-known series of analytical approximating functions $f_n(x) := exp\left(-\left(\frac{x-(a+b)/2}{(b-a)/4}\right)^{2n}\right)$ whose limit for $n \to \infty$ is equal to $\tilde{\chi}_{[a,b]}(x) = \begin{cases} 1 & \text{for } x \in (a,b) \\ 1/e & \text{for } x \in \{a,b\} \\ 0 & \text{for } x \in \mathbb{R}\backslash[a,b] \end{cases}$, a function differing from $\chi_{[a,b]}(x)$ on a set of measure zero only.

We may use these rectangle functions to approximate sufficiently smooth functions $f : \mathbb{R} \to \mathbb{R}$ for $x \in [a,b]$ by step functions as follows: Divide $[a,b]$ into m subintervals $[a_{k-1}, a_k]$ $(1 \leq k \leq m)$ of equal length $\Delta x = (b-a)/m$, i.e. $a_k = a + k(b-a)/m$. Then f can be roughly approximated by $\varphi(x) = \sum_{k=1}^{m} f(a_{k-1} + \vartheta_k \Delta x)\chi_{[a_{k-1},a_k]}(x)$ with certain $\vartheta_k \in (0,1)$. Using the beforementioned approximations, we obtain $\varphi(x) \approx \sum_{k=1}^{m} f(a_{k-1} + \vartheta_k \Delta x) \exp\left\{-\left(\frac{x-(a_{k-1}+a_k)/2}{\Delta x/2}\right)^{2n}\right\}$ for sufficiently large n. The same rectangle function may be used to construct arbitrary functions by replacing the coefficients $f(x_k - \vartheta\Delta x)$ with constants A_k.

4 Extension to Two and Three Dimensions

The characteristic function of rectangular parallelepipeds $C_k := \times_{i=1}^{k}[a_i, b_i]$ in two $(k = 2)$ and three $(k = 3)$ dimensions may be written $\chi_{C_k}(\vec{x}) := \prod_{i=1}^{k} \chi_{[a_i,b_i]}(x_i)$. The extension to triangles and 3-simplices (tetrahedrons), defined by their respective vertices $\vec{x}_i = (x_{i1}, \ldots, x_{ik})$ $(1 \leq i \leq (k+1)$ where k is the dimension, may be written $\chi_T(\vec{x}) = \prod_{1 \leq i < j \leq k+1} \chi_{[0,1]}\left(\frac{(\vec{x}-\vec{x}_i)\cdot(\vec{x}_j-\vec{x}_i)}{|\vec{x}_j-\vec{x}_i|^2}\right)$.

5 Simplified Model System

For the sake of simplicity, we are going to consider an extremely simple system composed of a cubic vessel of volume V_V floating in the center of a water-filled basin of square cross section, A. Let m denote the mass of the vessel and assume that the volume of water within the basin equals V_w. The depth of water in the basin is therefore equal to $h = V_w/A + m/(A\rho_{H_2O})$. Throwing some cargo of mass Δm into the water at time t_1 will result in a reduction of the vessels mass and a modification of the resulting depth of the water if the cargo's density, ρ_C, is greater than the density of water: $\rho_C > \rho_{H_2O}$. The new depth is equal to $h' = V_w/A + (m - \Delta m)/(A\rho_{H_2O}) + \Delta m/(A\rho_C)$. By the way, this is less than h.
If we are solely interested in the density distribution within this system, we may subdivide the vessel and the basin into rectangles and determine the resulting density function prior to putting the cargo into the water as follows:

$$\begin{aligned}\rho_0(\vec{x}) &= m/a^3 \cdot \chi_{C_V}(\vec{x}) + \rho_{H_2O} \cdot (\chi_{[0,b]\times[0,b]\times[0,h]}(\vec{x}) \\ &- \chi_{[(b-a)/2,(b+a)/2]\times[(b-a)/2,(b+a)/2]\times[h-m/(a^2\rho_{H_2O}),h]}(\vec{x}))\end{aligned}$$

with $C_V = [(b-a)/2, (b+a)/2] \times [(b-a)/2, (b+a)/2] \times [h - m/(a^2\rho_{H_2O}), h - m/(a^2\rho_{H_2O}) + a]$. In this relation, $b = A^{1/2}$ and $a = V_V^{1/3}$. Assuming that the cargo be evenly distributed on the basin's floor, with a layer thickness of $\delta = \Delta m/(\rho_C b^2)$, the modified density distribution is then given by

$$\begin{aligned}\rho_1(\vec{x}) &= \rho_C \chi_{[0,b]\times[0,b]\times[0,\delta]}(\vec{x}) + \rho_{H_2O} \cdot (\chi_{[0,b]\times[0,b]\times[\delta,h']}(\vec{x}) \\ &- \chi_{[(b-a)/2,(b+a)/2]\times[(b-a)/2,(b+a)/2]\times[h'-(m-\Delta m)/(a^2\rho_{H_2O}),h']}(\vec{x})) \\ &+ (m-\Delta m)/a^3 \cdot \chi_{C_{V'}}(\vec{x})\end{aligned}$$

with

$$\begin{aligned}C_{V'} &= [(b-a)/2, (b+a)/2] \times [(b-a)/2, (b+a)/2] \\ &\quad \times [h' - (m-\Delta m)/(a^2\rho_{H_2O}), h' - (m-\Delta m)/(a^2\rho_{H_2O}) + a]\end{aligned}$$

The time, t_1 at which the cargo is being put into the water may depend on some internal states of the vessel. Disregarding the dynamics of the movement, we may thus approximately describe the evolution of the system by

$$\rho(\vec{x}, t) = \rho_0(\vec{x})\,(1 - \Theta(t - t_1)) + \rho_1(\vec{x})\Theta(t - t_1)$$

Considering only finite times in an interval $t_0 < t_1 < t_m$, we may write $\rho(\vec{x}, t) = \rho_0(\vec{x})\chi_{[t_0,t_1]}(t) + \rho_1(\vec{x})\chi_{[t_1,t_m]}(t)$.

6 Extension to Complex Engineering Systems

The beforementioned techniques may readily be extended to describe more complex systems that cannot easily be described by differential equations or some open systems whose boundary conditions are time-dependent. At this stage, we did not present the set of tools necessary to do this in a coherent and satisfying manner. Future articles will fill this gap, describing modeling aspects of the evolution of computer controlled systems, like actuators in a set of car brakes or the motion of robot arms in manufacturing. Another standard example is furnished by the approximate description of fluid flow in a system of tubes that are interconnected with externally controlled valves. It will be shown, by way of example, that the accuracy of the qualitative modeling process can be increased by replacing the beforementioned step function approximation by suitably chosen polynomials that are again defined on the characteristic function of the system's subparts. These techniques, similar to using shape functions but in other parts not equal to the methodology in FEM calculations, will be described in a future article.

References

[1] Y. Choquet-Bruhat, C. DeWitt-Morette: *Analysis, Manifolds and Physics*, Revised edn (North-Holland, Amsterdam New York Oxford 1982)